15

ON ZERO MASS MESON-MESON SCATTERING

B. DEO

Physics Department, Ravenshaw College, Cuttack

(*Received, December*, 10, 1959)

ABSTRACT. The forward scattering matrix elements for the scattering of two mesons (zero mass) has been calculated from pair reproduction cross section by the method of analytic continuation. This has been compared with the Feynman matrix elements to the required order. By comparing the results of the two methods, the value of the counter $\lambda\phi^4$ term has been evaluated.

INTRODUCTION

Besides primitive divergencies, there is a class of divergences (comparable to photon-photon scattering divergency) peculiar to meson-nucleon interactions. The degree of divergence D for the process is determined by the well known relation

$$D = 4 - 3/2\ F_e - B_e$$

where F_e = number of external fermion lines

Be = number of external Boson lines.

Thus closed loops of fermions with three and four vertices of bosons are divergent. For photons and pseudoscalar mesons which are of importance in physics, a closed loop with three vertices have zero matrix element. But the four-vertex diagram is permissible and also possible giving rise to processes like photon-photon or meson-meson scattering. Therefore, suitable counter term (i) $\lambda A_\mu{}^4$ or (ii) $\lambda\phi^4$, where A_μ is the photon and ϕ the meson field operator, are to be added to the interaction Hamiltonian. For electrodynamics, gauge invariance forbids the occurence of the term $\lambda A_\mu{}^4$ in the Lagrangian; as a consequence, terms proportional to it can be dropped since they are non-gauge-invariant. Fortunately, however, if one calculates the matrix elements there is no divergence if the contribution for all the three basic diagrams are added up, as shown by Jauch and Rohrlich (1955)

But for meson-meson scattering, gauge invariance is not available. This is the first (and only) divergent process which is not eliminated by a renormalisation of mass or coupling constant in pseudoscalar theory. Therefore a counter term $\lambda\phi^4$ is essential. However the exact value of λ even in the lowest order has not been calculated since no method exists as to its unambiguous evaluation, as has been pointed out by Schweber, Bethe and de' Hoffmann (1955). Therefore, it is interesting to be able to complete this counter term to see to what extent the absence of gauge invariance can be compensated by alternate mode of thought.

FORMULATION OF THE PROBLEM

We shall use natural units $\hbar = c = 1$.

Let a pseudoscalar boson field ϕ and a fermi field ψ interact. the Hamiltonian in interaction representation being given by,

$$H\mathrm{int} = ig \int d^3x \, \bar{\Psi}\gamma_5\psi\phi$$

The fourth-order term of S-matrix which describes the effects under consideration is given by the Feynman diagram. (Fig. 1)

or by the integral

$$S^{(4)} = -\frac{1}{64} g^4 \int\int\int\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$$

$$T_r\{\gamma_5 \; S_F(x_2-x_1)\gamma_5 S_F(x_3-x_2)\gamma_5 \; S_F(x_4-x_3)\gamma_5 \; S_F(x_1-x_4)\} \qquad \ldots \quad (1)$$

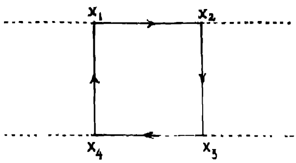

Fig. 1

$S_F(x)$ being the propagation function of the fermi field (mass $= m$)

$$S_F(x) = -\frac{2i}{(2\pi)^4} \underset{\epsilon \to 0^+}{\mathrm{Lt}} + \int d^4p \,.. \frac{i\gamma \,.\, p - m}{p^2 + m^2 - i\epsilon} e^{-ip\,x}$$

$$p \,.\, x = p \,.\, r - p_0 x_0.$$

The transition to momentum space with

$$\phi(x) = \int \phi(k) e^{ik\,x} d^4k \qquad \ldots \quad (1a)$$

yields,

$$S^{(4)} = -\frac{i}{4}\left(\frac{g^2}{4\pi}\right)^2 \int d^4x \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 \; \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4).$$

$$.e^{\,i(k_1+k_2+k_3+k_4)\,.\,x} \frac{1}{3} \, G(k_1k_2k_3k_4) \qquad \ldots \quad (2)$$

where

$$G(1234) = T^{(1)}(1234) + T^{(2)}(1, 2, 4, 3) + T^{(3)}(1, 3, 24) \qquad \ldots \quad (2a)$$

and

$$T^{(1)}(1234) = \frac{1}{i\pi^2} \int d^4p \,.\, Tr.\{\gamma_5 S_F(p)\gamma_5 S_F(p-k_2)\gamma_5 S_F(p-k_2-k_3)\gamma_5 S_F (p-k_2-k_3-k_4)\}$$

$$T^{(2)} = T^{(1)}(k_2 \Longleftrightarrow k_4)$$

$$T^{(3)} = T^{(1)}(k_2 \Longleftrightarrow k_3) \qquad \ldots \quad (3)$$

It is easily seen that the divergent term is entirely contained in the terms proportional to $(p^2)^2$ in the numerator and is logarithmically divergent. Therefore subtraction any term whatsoever, containing the log-divergent term shall eliminate this divergence.

To get round this difficulty we first note that the quantity we wish to subtract is essentially a constant independent of the scattering angle. So our problem will be considerably simplified if we can compute the matrix elements for forward scattering only We let,

$$k_1 = k; \quad k_3 = -q; \quad k_2 = -k; \quad k_4 = q$$

Then only invariant that can be constructed is $k.q$, and in a c.m. system, $k.q = -2\omega^2$ for forward scattering. So we write,

$$G(k, q) = G(k.q) = G(\omega^2, 0) = G(t, 0) \qquad \ldots \quad (4)$$

where $t = w^2/m^2$ for convenience.

The matrix element is a function of the invariant 't' $= -2k \,.\, q/4m^2$. We can further separate the real and imaginary part of the Function

$$G(t, 0) = a_1(t) + ia_2(t) \qquad \ldots \quad (5)$$

The imaginary part $a_2(t)$ is related to the pair production cross-section.

The unitarity of S-matrix, makes it possible to write the total transition probability for the production of a pair of fermions by two messons,

$$P = -2R_e[\psi_i, \; S^{(4)}\psi_i]$$

$$= -\left[\frac{2}{V^2}\int d^4x\right]\left(\frac{g^2}{4\pi}\right)^2 \frac{1}{t} \text{ Im. } G(t, 0) \qquad \ldots \quad (6)$$

This is evaluated by the usual substitution, (See reference R. Karplus and M. Neuman (1951)).

$$\phi(\vec{k}, k_0) \rightarrow V(2\,|\,k\,|\,V)^{-\frac{1}{2}}[\vec{a(k)}\delta(k_0 - |\vec{k}|) + a^{\dagger}(-\vec{k})\delta(k_0 + |\vec{k}|)$$

$$\int \vec{dk} \rightarrow \frac{1}{V}\sum_k$$

where $a(\vec{k})$ and $a^\dagger(-\vec{k})$ are the annihilation and creation operators and V a large but finite volume. The total cross-section of pair production by two mesons of momentum

$$\vec{k}_1 = \vec{k}\,;\quad \vec{k}_2 = -\vec{k},\quad |\vec{k}_1| = |\vec{k}_2| = \omega \qquad \text{is then}$$

$$\sigma_{pair}(t) = 2(g^2/4\pi)^2 \frac{1}{m^2 t} \operatorname{Im} G(t, 0)$$

Thus

$$a_2(t) = \frac{1}{(g^2/4\pi)^2} \frac{m^2 t}{2} \sigma_{pair}(t) \qquad \ldots \quad (7)$$

Following the same procedure, the probability for meson-meson scattering is

$$P = \left[\frac{2}{V^2} \int d^4x \right] \left(g^2/4\pi \right)^4 \frac{1}{128\pi^2} \frac{1}{m^2 t} \int d\Omega(\vec{k}) \; |G(t)|^2 \qquad \ldots \quad (8)$$

when the diff. scattering cross-section in the forward direction in c.m. system is

$$\frac{d\sigma}{d\Omega}(t, 0) = \left(\frac{g^2}{4\pi} \right)^4 \frac{1}{64\pi^2 m^2 t} \; |G(t, 0)|^2$$

$$= \left(\frac{g^2}{4\pi} \right)^4 \frac{1}{64\pi^2 m^2 t} \; |a_1 + ia_2|^2 \qquad \ldots \quad (9)$$

We see that pair production cross-section $\sigma_{pair}(t)$ shall enable us to write $a_2(0, t)$. We shall then resort to the method of analytic continuation, suggested by Toll (1952) and calculate $a_2(t, 0)$.

PAIR PRODUCTION CROSS-SECTION

The matrix element for pair production is

$$\langle pp' \mid M \mid k_1 k_2 \rangle = \frac{ig^2}{(2\pi)^2} \frac{m}{\sqrt{\epsilon_p \epsilon_{p'} 2\omega_1 2\omega_2}}\,.$$

$$.\bar{u}(p) \left[\gamma_5 \frac{i\gamma \,.\, \overline{p - k_1} - m}{2p \,.\, k_1} \gamma_5 + \gamma_5 \frac{i\gamma \,.\, \overline{p - k_2} - m}{2p \,.\, k_2} \gamma_5 \right] v(p') \qquad \ldots \quad (10)$$

arising from the crossed and uncrossed Feynman diagrams. The total scattering cross-section can be evaluated in standard way and one gets,

$$\sigma_{pair}(t) = 8\pi \left(\frac{g^2}{4\pi}\right)^2 \frac{1}{m^2 t} \left[\log \frac{1 + \left(1 - \frac{1}{t}\right)^{\frac{1}{2}}}{1 - \left(1 - \frac{1}{t}\right)^{\frac{1}{2}}} - 2\left(1 - \frac{1}{t}\right)^{\frac{1}{2}} \right], \; t > 1$$

$$= 0 \qquad t < 1 \tag{11}$$

Equation (7) now gives

$$a_2(t) = -4\pi \left[\log \frac{1 + \left(1 - \frac{1}{t}\right)^{1/2}}{1 - \left(1 - \frac{1}{t}\right)^{1/2}} - 2\left(1 - \frac{1}{t}\right)^{1/2} \right], \; t > 1$$

$$= 0 \;, \; t < 1 \tag{12}$$

However, these expressions hold good only for positive and real t.

For the use of analytic continuation we must know the value of Im. $G(t, 0)$ for $t < 0$. To find this out we resort to the Feynman diagrams as depicted in Figs 2(a), 2(b), and 2(c) which gives respectively $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ of equation 2(a) with $k_1 = -k_2 = k$, $k_3 = -k_4 = q$

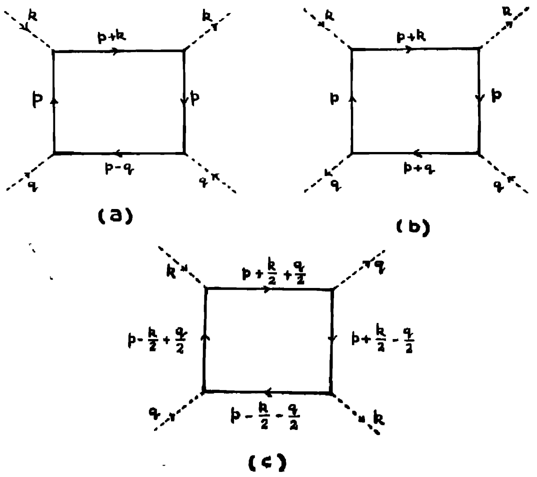

Fig. 2

From general four-momentum conservation we label the integral fermion lines as shown. Figure 2(c) we have shifted the value of 'p' which does not bring in any surface term, since the matrix element is only log. divergent. Inspection of Figs. 2(a) and 2(b) shows that

$$T^{(1)} = T^{(1)}(k.q) \tag{13a}$$

$$T^{(2)} = T^{(1)}(k,-q) = T^{(1)}(-k\,.q) \tag{13b}$$

$$T^{(1)} + T^{(2)} = T^{(1)}(k.q) + T^{(1)}(-k.q) \tag{13c}$$

This means that the matrix elements of $T^{(2)}(k, q)$ can be obtained from $T^{(1)}(k, q)$ by replacing q by $-q$. Since the contribution from each diagram must be a function of the only available invariant $k\,q$, equations 13(b) and 13(c) follows.

In diagram, Fig. 2(c), we note immediately that,

$$T^{(3)}(k, q) = T^{(3)}(k, -q) = T^{(3)}(-k, q) \tag{14}$$

Thus

$$G(t, 0) = G(-t, 0) \tag{15}$$

These symmetry properties are adequate to calculate the Matrix elements for forward scattering from pair production cross-section by analytic continuation. The S-matrix matrix elements can also be computed directly by Feynman method. The results from these two methods shall be compared. Retaining only those terms that can be obtained from analytic continuation, the counter renormalisation term for a finite S-matrix can be found out. We shall first make the necessary analytic continuation, in the next section and in the succeeding section, find the S-matrix elements.

METHOD OF ANALYTIC CONTINUATION

Following Rohrlich and Gluckstern (1952), we consider Cauchy's Theorem,

$$f(x) = \frac{1}{2\pi i} \oint \frac{f(z')}{z'-x}\, dz' \qquad \ldots \quad (16)$$

Let z be on the real axis, $z = t$ and assume that $f(z)$ is regular for Im.$(z) > -\epsilon$; $\epsilon > 0$ We can choose the path of integration as shown in Fig. 3 and write

$$f(t) = \frac{1}{2\pi i}\, P \int_{-R}^{+R} \frac{f(t')}{t'-t}\, dt' + \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{-} \frac{f(z')}{z'}\, dz' \qquad \ldots \quad (17)$$

where in the last integral t has been neglected in the denominator, since for the contour $R; |z'| \gg t$ everywhere. We further assume that $f(z')/z'$ is regular at $z'=0$, so that

$$\frac{f(z')}{z'}\, dz' = -\int \frac{f(t')}{t'}\, dt' \qquad \dots (18)$$

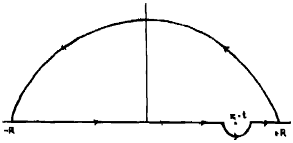

Fig. 3

Separating $f(t)$ into its real and imaginary parts, we obtain with $R \to \infty$

$$\mathrm{Re}\, f(t) = \frac{t}{\pi} P \int_{-\infty}^{+\infty} \frac{Im\ f(t')}{t'(t'-t)}\, dt' \qquad (19)$$

We identify $f(t)$ with $G(0,t)$ and since,

$$G(0,t) = G(0-t)$$

we get

$$Re\ G(0,t) = 2t^2 \frac{P}{\pi} \int_q^{\infty} \frac{\mathrm{Im}\ .\ G(0,t)}{t'^2-t^2} \frac{dt'}{t'} \qquad \dots (20)$$

Thus we get

$$a_1(t) = -8t^2 P \int_1^{\infty} \left[\log \frac{1+\left(1-\frac{1}{t'}\right)^{\frac{1}{2}}}{1-\left(1-\frac{1}{t'}\right)^{\frac{1}{2}}} - 2\left(1-\frac{1}{t'}\right)^{\frac{1}{2}} \right] \frac{dt'}{t'} \cdot \frac{1}{t'^2-t^2}$$

$$= 8t^2 P \int_0^1 \frac{p'dp'}{1-t^2p'^2} \left[\log \frac{1-(1-p')^{\frac{1}{2}}}{1+(1-p')^{\frac{1}{2}}} + 2(1-p')^{\frac{1}{2}} \right] \qquad \dots (21)$$

where we have put $t' = \dfrac{1}{p'}$

These integrals can be evaluated in straight forward way. But for sake of comparison, we shall make the following substitutions.

Integrating by parts,

$$a_1(t) = 4P \int_0^1 \left(\frac{1}{(1-p')^{\frac{1}{2}} p'} - \frac{1}{(1-p')^{\frac{1}{2}}} \right) \log\,(1-t^2p'^2)dp'$$

Letting

$$\sqrt{1-p'} = u, \; -\frac{1}{2} \frac{dp'}{\sqrt{1-p'}} = du$$

$$a_1(t) = 8P \int_{0}^{1} \left(\frac{1}{1-u^2} - 1 \right) \log \left[1 - 16t^2 \left(\frac{1-u^2}{4} \right)^2 \right] du$$

$$= 4P \int_{-1}^{+1} \left(\frac{1}{1-u^2} - 1 \right) \log \left[1 - 16t^2 \left(\frac{1-u^2}{4} \right)^2 \right] du$$

$$= 8P \int_0^1 \left(\frac{1}{4x} - 1 \right) [\log\,(1-4tx(1-x)) + \log\,(1+4tx(1-x))]$$

$$\text{with} \qquad x = \frac{1+u}{2}, \qquad \qquad (22)$$

Evaluating

$$\frac{1}{8}\, a_1(0,\omega) = \left(\sinh^{-1} \frac{\omega}{m} \right)^2 - 2 \left(1 + \frac{m}{\omega} \right)^{\frac{1}{2}} \sinh^{-1} \frac{\omega}{m}$$

$$+ \begin{cases} -\left(\sin^{-1} \frac{\omega}{m} \right)^2 - 2 \left(\frac{m^2}{\omega^2} - 1 \right)^{\frac{1}{2}} \sin^{-1} \frac{\omega}{m}\,; & 0 < \omega < m \\ \left(\cosh^{-1} \frac{\omega}{m} \right)^2 - \frac{\pi^2}{4} - 2 \left(1 - \frac{m^2}{\omega^2} \right)^{\frac{1}{2}} \cosh^{-1} \frac{\omega}{m}\,; & \omega > m. \end{cases} \qquad \ldots \quad (23)$$

$$\frac{1}{8}\, a_2(0,\omega) = -\pi \left\{ \cosh^{-1} \frac{\omega}{m} - \left(1 - \frac{m^2}{\omega^2} \right)^{\frac{1}{2}} \right\} \,; \quad \omega > m$$

$$= 0 \;\; ; \;\; \omega < m. \qquad \ldots \quad (24)$$

Using equation (9), one can now calculate meson-meson scattering cross-section in the forward direction. This is what a physicist would have done, had Feynman method not been known, analytic contuation methods were known.

DIRECT EVALUATION OF MATRIX ELEMENTS

One now proceeds to calculate directly Matrix elements $G(kq)$ of equation (2), using the Feynman's diagrams Figs. 2(a), (b) and (c). Even in case of forward scattering, the calculations are extremely lengthy and tedious. The proceedure and details of calculation have been outlined by Jost, Luttinger and Slotnick (1950). We merely quote the results, (the details of calculations can be supplied on request),

$$G(k_1q) = 12D_\infty + 8\int_0^1 \left(\frac{1}{4y} - 1\right) \log\left(1 - \frac{2kq}{m^2}\, y(1-y)\right)$$

$$+ 8\int_0^1 \left(\frac{1}{4y} - 1\right) \log\left(1 + \frac{2kq}{m^2}\, y(1-y)\right)$$

where $$D_\infty = \frac{1}{i\pi^2}\int d^4p \, . \frac{(p^2)^2 + 2m^2p^2 + m^4}{(p^2+m^2)^4} \qquad (25)$$

Letting $2k\,q = -4\omega^2 = -4m^2t$, we see that $G(k,q) - 12D_\infty = G_f$ gives correctly the values of a_1 and a_2 deduced before. Referring to equation (2) we see that, in general,

$$S^{(4)} = -\frac{i}{12}\left(\frac{g^2}{4\pi}\right)^2 \int d^4x \int d^4k_1 \ldots d^4k_1 \; e^{i(k_1+k_2+k_3+k_4)\,.\,x} \; \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)$$

$$[12D_\infty + G_f(k_1\,k_2\,k_3\,k_4)] \qquad \ldots \quad (26)$$

The first term (which is diverent) is $-i\left(\frac{g^2}{6\pi}\right)^2 D_\infty \int d^4x\phi^4(x)$ since the momentum space transform is given by (1a).

Thus if we start with the interaction Hamiltonian density

$$H_{int} = ig\bar{\psi}\gamma_5\psi\phi - \left(\frac{g^2}{4\pi}\right)^2 D_\infty\phi^4(x) \qquad .. \quad (27)$$

then, the contribution due to the second term in the first order is

$$\frac{(-i)^1}{\lfloor 1} \left(\frac{g^2}{4\pi}\right)^2 D_\infty \int \phi^4(x)d^4x = +i\left(\frac{g^2}{4\pi}\right) D_\infty \int d^4x\phi^4(x)$$

which cancels the first term in (26). So the finite $S^{(4)}$ is now given by

$$S^{(4)} = -\frac{i}{12}\left(\frac{g^2}{4\pi}\right)^2 \int d^4x \int \;\; .. \quad G_f(k_1\,k_2\,k_3\,k_4)$$

This G_f in the forward direction shall now give

$$G_f(0, t) = a_1 + ia_2$$

which can be derived by analytic continuation method.

Thus the contact renormalisation term $\lambda\phi^4$ that must be subtracted from the interaction Hamiltonian density is unambiguously determined with

$$\lambda = \left(\frac{g^2}{4\pi}\right)^2 \frac{1}{i\pi^2} \int \frac{d^4p}{(p^2+m^2-i\epsilon)^4} \left((p^2)^2+2m^2p^2+m^4\right)$$

$$= \left(\frac{g^2}{4\pi}\right)^2 \left[\underset{\Delta\to\infty}{\alpha t} \log \frac{\Delta^2+m^2}{m^2} - 1\right] \quad \ldots$$

The low energy theorem then turns out to be

$$\underset{t\to 0}{\alpha t}\ G(0, t) = 0$$

We have used the method of analytic continuation which can also be derived from dispersion relations Thus it seems that the causality requirements may as well provide the clue for a low energy theorem for meson-meson scattering. An application of the general dispersion relation to evaluate the low energy limit for $\pi-\pi$ scattering is in progress.

ACKNOWLEDGMENTS

Author's thanks are due to Dr. T. Pradhan and Shri S. P. Misra for many helpful discussions.

REFERENCES

Jauch, J. M. and Rohrlich, F., 1955, The theory of Electrons and photons, Addison and Wesley Publishing Co.

Jost, R., Luttinger, J. and Slotnick, M., 1950, *Phys. Rev.*, **80**, 189.

Karplus, R. and Neuman, M., 1951, *Phys. Rev.*, **83**, 776.

Rohrlich, F. and Gluckstern, R., 1952, *Phys. Rev.*, **86**, 1.

Schweber, S. S., Bethe, H. A. and de Hoffman, F., 1955, Mesons and Fields, Vol. I, Row Peterson & Co.

Toll, J. S., 1952, Thesis, Princeton University.